LIE SYSTEMS, LIE SYMMETRIES AND RECIPROCAL TRANSFORMATIONS

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Outline

LIE SYSTEMS

1. Their solutions in form of superposition rules and through the Lie group approach.
2. Lie systems on different geometries: *structure-Lie systems.*
3. Classification of structure-Lie systems and examples of mathematical and physical interest and in other disciplines: biology, medicine.

LIE SYMMETRIES

1. Lie symmetries of Lie systems.
2. Lie symmetries and reduction of nonlinear hierarchies of PDEs in $2+1$ dimensions in the realm of Fluid Mechanics.
3. Lie symmetries and reduction of Lax pairs in $2+1$ dimensions (associated with nonlinear hierarchies).

MIURA-RECIPROCAL TRANSFORMATIONS

1. Construction of reciprocal transformations of nonlinear PDEs.
2. Construction of Miura-reciprocal transformations linking hierarchies of PDEs.
3. A method to derive Lax pairs by means of reciprocal trasformations.

PART I: LIE SYSTEMS

1. Lie systems: characterization and solutions. *Superposition rules and the Lie group approach.*
2. Lie–Hamilton systems: classification on $\mathbb{R}^2$. *Applications to systems of mathematical and physical relevance. Superposition rules with the coalgebra method.*
3. Dirac–Lie systems: *Dirac–Lie Hamiltonians and construction of bi-Dirac–Lie systems.*
4. Jacobi–Lie systems: *Jacobi–Lie Hamiltonians and their classification on the plane.*

Lie systems: characterization and solutions

Lie systems admit general solutions in form of (generally non-linear) **superposition rules** or functions $\Phi : N^m \times N \to N$ of the form $x = \Phi(x_{(1)}, \ldots, x_{(m)}; k)$ allowing us to write the general solution as

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k),$$

where $x_{(1)}(t), \ldots, x_{(m)}(t)$ is a generic family of particular solutions and $k \in N$.

Theorem

A system X_t is a Lie system and consequently admits a superposition rule if and only if $X_t = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha$ spans an r-dimensional Lie algebra of vector fields, $X_1, \ldots, X_r$ the so-called **Vessiot–Guldberg Lie algebra** *(VG) associated with X_t, for certain functions $b_1(t), \ldots, b_r(t)$.*

The SORE

Second-order Riccati equations (SORE) are given by the family

$$\frac{d^2x}{dt^2}+(f_0(t)+f_1(t)x)\frac{dx}{dt}+c_0(t)+c_1(t)x+c_2(t)x^2+c_3(t)x^3=0,$$

with $\quad f_1(t)=3\sqrt{c_3(t)}, \qquad f_0(t)=\frac{c_2(t)}{\sqrt{c_3(t)}}-\frac{1}{2c_3(t)}\frac{dc_3}{dt}(t), \qquad c_3(t)\neq 0$

They have a Hamiltonian $h(t,x,p)=p\left(\frac{1}{\sqrt{-p}}-U(t,x)\right)-\sqrt{-p}=-2\sqrt{-p}-p\,U(t,x)$. with $U(t,x)=a_0(t)+a_1(t)x+a_2(t)x^2$.

$$\frac{dx}{dt}=\frac{\partial h}{\partial p}=\frac{1}{\sqrt{-p}}-U(t,x)=\frac{1}{\sqrt{-p}}-a_0(t)-a_1(t)x-a_2(t)x^2,$$
$$\frac{dp}{dt}=-\frac{\partial h}{\partial x}=p\frac{\partial U}{\partial x}(t,x)=p(a_1(t)+2a_2(t)x).$$

expressible in terms of a t-dependent vector field $X_t=X_1-a_0(t)X_2-a_1(t)X_3-a_2(t)X_4$.

$$X_1=\frac{1}{\sqrt{-p}}\frac{\partial}{\partial x},\quad X_2=\frac{\partial}{\partial x},\quad X_3=x\frac{\partial}{\partial x}-p\frac{\partial}{\partial p},$$
$$X_4=x^2\frac{\partial}{\partial x}-2xp\frac{\partial}{\partial p},\quad X_5=\frac{x}{\sqrt{-p}}\frac{\partial}{\partial x}+2\sqrt{-p}\frac{\partial}{\partial p},$$

that span a five-dimensional VG. Then, *SORE are a Lie system*.

Some examples of Lie systems

- The **matrix Riccati equations** on $\mathbb{R}^2$ with $V^{\rm MR}$ which is isomorphic to $\mathfrak{sl}(3,\mathbb{R})$.
- The **Milne–Pinney equations** have a VG isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.
- **Cayley–Klein Riccati equations** with $z \in \{\mathbb{C}, \mathbb{C}', \mathbb{D}\}$ have VG isomorphic to $\mathfrak{sl}(2)$
- The **generalized Buchdahl equations** have a VG isomorphic to $\mathfrak{h}_2$.
- The ***t*-dependent Lotka–Volterra system** has a VG isomorphic to $\mathfrak{h}_2$.
- The **complex Bernoulli equation** with t-dependent real coefficients with $\mathfrak{h}_2$..
- The **dissipative harmonic oscillator** with VG isomorphic to $\mathfrak{sl}(2) \ltimes \mathbb{R}^2$.
- **Simple viral infection models** with VG isomorphic to $V \simeq \mathbb{R} \ltimes \mathbb{R}^2$.
- A **planar diffusion Riccati system** with VG $\mathfrak{sl}(2,\mathbb{R})$.

Superposition rules through an algorithmical method

Definition

Given a t-dependent vector field X on N, its **diagonal prolongation** $\widetilde{X}$ to $N^{(m+1)}$ is the unique t-dependent vector field on $N^{(m+1)}$ such that

- Given $\mathrm{pr} : (x_{(0)}, \ldots, x_{(m)}) \in N^{(m+1)} \mapsto x_{(0)} \in N$, we have that $\mathrm{pr}_* \widetilde{X}_t = X_t \ \forall t \in \mathbb{R}$.
- $\widetilde{X}$ is invariant under the permutations $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 0, \ldots, m$.

In coordinates, we have that

$$X_j = \sum_{i=1}^{n} X^i(t,x)\frac{\partial}{\partial x_i} \Rightarrow \widetilde{X} = \sum_{j=0}^{m+1} X_j = \sum_{j=0}^{m+1}\sum_{i=1}^{n} X^i(t,x)\frac{\partial}{\partial x_i}.$$

Given the basis of the VG, we choose the minimum integer m such that the diagonal prolongations to N^m are linearly independent. Then, obtain n functionally independent first integrals and assume they take constant values $F_i = k_i$. We can employ these equalities to express the variables $(x_1)_{(0)}, \ldots, (x_n)_{(0)}$ in terms of the variables of the other copies of N within $N^{(m+1)}$ and the constants $k_1, \ldots, k_n$.

Second-order Riccati equation

The VG is isomorphic to $V \simeq V_1 \oplus_s V_2$, with $V_2 = \langle X_2, X_3, X_4 \rangle$ being a semisimple Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and $V_1 = \langle X_1, X_5 \rangle$ is the radical.

To obtain a superposition rule we need two common functionally independent first-integrals I_1, I_2, for $\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3, \widetilde{X}_4, \widetilde{X}_5$ to a certain $\mathrm{T}^* N^{(m+1)}$.

In our case, it can easily be verified that $m = 3$. These are

$$F_0 = (x_{(2)} - x_{(3)})\sqrt{p_{(2)}p_{(3)}} + (x_{(3)} - x_{(1)})\sqrt{p_{(3)}p_{(1)}} + (x_{(1)} - x_{(2)})\sqrt{p_{(1)}p_{(2)}} = k_3,$$

If $k_1 = (F_0)_{012}$ and $k_2 = (F_0)_{013}$ where k_1, k_2 are real constants, then

$$x_{(0)} = \frac{k_1\Gamma(x_{(1)}, p_{(1)}, x_{(3)}, p_{(3)}) + k_2\Gamma(x_{(2)}, p_{(2)}, x_{(1)}, p_{(1)}) - F_0 x_{(1)}\sqrt{-p_{(1)}}}{k_1(\sqrt{-p_{(1)}} - \sqrt{-p_{(3)}}) + k_2(\sqrt{-p_{(2)}} - \sqrt{-p_{(1)}}) - \sqrt{-p_{(1)}}F_0},$$

$$p_{(0)} = -\left[k_1/F_0(\sqrt{-p_{(3)}} - \sqrt{-p_{(1)}}) + k_2/F_0(\sqrt{-p_{(1)}} - \sqrt{-p_{(2)}}) + \sqrt{-p_{(1)}}\right]^2,$$

where $\Gamma(x_{(i)}, p_{(i)}, x_{(j)}, p_{(j)}) = \sqrt{-p_{(i)}}x_{(i)} - \sqrt{-p_{(j)}}x_{(j)}$.

Lie systems on Lie groups

Every Lie system X associated with a VG gives rise to a (generally local) Lie group action $\varphi : G \times N \to N$ whose fundamental vector fields are $T_eG \simeq V$.
We write the general solution of X as

$$x(t) = \varphi(g_1(t), x_0), \qquad x_0 \in N,$$

with $g_1(t)$ being a particular solution of

$$\frac{dg}{dt} = -\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^R(g),$$

where $X_1^R, \ldots, X_r^R$ is a certain basis of right-invariant vector fields on G such that $X_\alpha^R(e) = \mathrm{a}_\alpha \in T_eG$, with $\alpha = 1, \ldots, r$, and each a_α is the element of T_eG associated with the fundamental vector field X_α.

2nd-order Kummer–Schwarz equations

Consider the first-order system associated with the **second-order Kummer–Schwarz equations (SOKSE)** on $\mathrm{T}\mathbb{R}_0$, with $\mathbb{R}_0 = \mathbb{R} - \{0\}$

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = \frac{3}{2}\frac{v^2}{x} - 2c_0x^3 + 2b_1(t)x,$$

This system describes the integral curves of the t-dependent vector field

$$X_t = v\frac{\partial}{\partial x} + \left(\frac{3}{2}\frac{v^2}{x} - 2c_0x^3 + 2b_1(t)x\right)\frac{\partial}{\partial v} = M_3 + b_1(t)M_1,$$

where

$$M_1 = 2x\frac{\partial}{\partial v}, \qquad M_2 = x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}, \qquad M_3 = v\frac{\partial}{\partial x} + \left(\frac{3}{2}\frac{v^2}{x} - 2c_0x^3\right)\frac{\partial}{\partial v}$$

satisfy the commutation relations

$$[M_1, M_3] = 2M_2, \quad [M_1, M_2] = M_1, \quad [M_2, M_3] = M_3.$$

These vector fields span a VG *isomorphic to* $\mathfrak{sl}(2, \mathbb{R})$.

2nd-order Kummer–Schwarz equations

Consider a Lie group action $\varphi_{2KS} : G \times \mathrm{T}\mathbb{R}_0 \to \mathrm{T}\mathbb{R}_0$ whose fundamental vector fields are V and $T_eG \simeq V$. Consider the basis of matrices of $\mathfrak{sl}(2,\mathbb{R})$ given by

$$\mathrm{a}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{a}_2 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{a}_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

satisfying the commutation relations $\quad [\mathrm{a}_1, \mathrm{a}_3] = 2\mathrm{a}_2, [\mathrm{a}_1, \mathrm{a}_2] = \mathrm{a}_1, [\mathrm{a}_2, \mathrm{a}_3] = \mathrm{a}_3.$

There exists a local Lie group action $\varphi_{2KS} : SL(2,\mathbb{R}) \times \mathrm{T}\mathbb{R}_0 \to \mathrm{T}\mathbb{R}_0$, $\mathbf{t}_x \equiv (x, v) \in \mathrm{T}_x\mathbb{R}_0 \subset \mathrm{T}\mathbb{R}_0$, $\alpha = 1, 2, 3$, and $s \in \mathbb{R}$.

$$\frac{d}{ds}\varphi_{2KS}(\exp(-s\mathrm{a}_\alpha), \mathbf{t}_x) = M_\alpha(\varphi_{2KS}(\exp(-s\mathrm{a}_\alpha), \mathbf{t}_x)),$$

Using the *canonical coordinates of the second kind*,

$$g = \exp(-\lambda_3\mathrm{a}_3)\exp(-\lambda_2\mathrm{a}_2)\exp(-\lambda_1\mathrm{a}_1),$$

and the standard matrix representation of $SL(2,\mathbb{R})$.

$$g \in SL(2,\mathbb{R}) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}.$$

2nd-order Kummer–Schwarz equations

By direct comparison,

$$\lambda_1 = -\beta/\alpha, \qquad \lambda_2 = 2\log\alpha, \qquad \lambda_3 = \gamma/\alpha,$$

the rewritten action reads

$$\varphi_{2KS}(g, \mathbf{t}_x) = \left(\frac{x}{F_g(\mathbf{t}_x)}, \frac{1}{F_g^2(\mathbf{t}_x)} \left[(v\alpha - 2x\beta)\left(\delta - \frac{\gamma v}{2x}\right) - 2c_0 x^3 \alpha\gamma \right] \right),$$

where

$$F_g(\mathbf{t}_x) = \left(\delta - \frac{\gamma v}{2x}\right)^2 + c_0 x^2 \gamma^2.$$

The action φ_{2KS} also permits us to write the general solution in the form $(x(t), v(t)) = \varphi_{2KS}(g(t), \mathbf{t}_x)$, with $g(t)$ being a particular solution of

$$\frac{dg}{dt} = -X_3^R(g) - b_1(t) X_1^R(g),$$

where X_α^R, with $\alpha = 1, 2, 3$, are the right-invariant vector fields on $SL(2, \mathbb{R})$ satisfying $X_\alpha^R(e) = \mathrm{a}_\alpha$.

Lie–Hamilton systems

Definition

A system X on N is a **Lie–Hamilton system (LH system)** if N can be endowed with a Poisson bivector Λ such that V^X becomes a finite-dimensional real Lie algebra of Hamiltonian vector fields with respect to Λ.

Consider again SORE in Hamiltonian form and the Poisson bivector $\Lambda = \partial/\partial x \wedge \partial/\partial p$ on $\mathcal{O}$. Note that $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$,

$$h_1(x,p) = -2\sqrt{-p}, \quad h_2(x,p) = p, \quad h_3(x,p) = xp,$$
$$h_4(x,p) = x^2 p, \quad h_5(x,p) = -2x\sqrt{-p}.$$

These functions span along with $\tilde{h}_0 = 1$ a Lie algebra of functions *isomorphic to the two-photon Lie algebra* $\mathfrak{h}_6$

$$\{h_1,h_3\}_\omega = -\frac{1}{2}\tilde{h}_1, \quad \{h_1,h_4\}_\omega = -h_5, \quad \{h_1,h_5\}_\omega = 2h_0,$$
$$\{h_2,h_3\}_\omega = -h_2, \quad \{h_2,h_4\}_\omega = -2h_3, \quad \{h_2,h_5\}_\omega = -h_1,$$
$$\{h_3,h_4\}_\omega = -h_4, \quad \{h_3,h_5\}_\omega = -\frac{1}{2}h_5.$$

Our classification of Lie–Hamilton systems on $\mathbb{R}^2$

#	Primitive	Hamiltonian functions h_i	ω	Lie–Hamilton algebra
P_1	$A_0 \simeq \mathfrak{iso}(2)$	$y,\ -x,\ \frac{1}{2}(x^2+y^2),\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{\mathfrak{iso}}(2)$
P_2	$\mathfrak{sl}(2)$	$-\dfrac{1}{y},\ -\dfrac{x}{y},\ -\dfrac{x^2+y^2}{y}$	$\dfrac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2) \oplus \mathbb{R}$
P_3	$\mathfrak{so}(3)$	$\dfrac{-1}{2(1+x^2+y^2)},\ \dfrac{y}{1+x^2+y^2},\ -\dfrac{x}{1+x^2+y^2},\ 1$	$\dfrac{\mathrm{d}x \wedge \mathrm{d}y}{(1+x^2+y^2)^2}$	$\mathfrak{so}(3)$ or $\mathfrak{so}(3) \oplus \mathbb{R}$
P_5	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$y,\ -x,\ xy,\ \frac{1}{2}y^2,\ -\frac{1}{2}x^2,\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{\mathfrak{sl}(2) \ltimes \mathbb{R}^2} \simeq \mathfrak{h}_6$

Our classification of Lie–Hamilton systems on $\mathbb{R}^2$

#	Imprimitive	Hamiltonian functions h_i	ω	Lie–Hamilton algebra
I_1	$\mathbb{R}$	$\int^y f(y')\mathrm{d}y'$	$f(y)\mathrm{d}x\wedge\mathrm{d}y$	$\mathbb{R}$ or $\mathbb{R}^2$
I_4	$\mathfrak{sl}(2)$ (type II)	$\dfrac{1}{x-y},\ \dfrac{x+y}{2(x-y)},\ \dfrac{xy}{x-y}$	$\dfrac{\mathrm{d}x\wedge\mathrm{d}y}{(x-y)^2}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2)\oplus\mathbb{R}$
I_5	$\mathfrak{sl}(2)$ (type III)	$-\dfrac{1}{2y^2},\ -\dfrac{x}{y^2},\ -\dfrac{x^2}{2y^2}$	$\dfrac{\mathrm{d}x\wedge\mathrm{d}y}{y^3}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2)\oplus\mathbb{R}$
I_8	$B_{-1}\simeq\mathfrak{iso}(1,1)$	$y,\ -x,\ xy,\ 1$	$\mathrm{d}x\wedge\mathrm{d}y$	$\overline{\mathfrak{iso}}(1,1)\simeq\mathfrak{h}_4$
I_{12}	$\mathbb{R}^{r+1}$	$-\int^x f(x')\mathrm{d}x', -\int^x f(x')\xi_j(x')\mathrm{d}x'$	$f(x)\mathrm{d}x\wedge\mathrm{d}y$	$\mathbb{R}^{r+1}$ or $\mathbb{R}^{r+2}$
I_{14A}	$\mathbb{R}\ltimes\mathbb{R}^r$ (type I)	$y,\ -\int^x\eta_j(x')\mathrm{d}x',\quad 1\notin\langle\eta_j\rangle$	$\mathrm{d}x\wedge\mathrm{d}y$	$\mathbb{R}\ltimes\mathbb{R}^r$ or $(\mathbb{R}\ltimes\mathbb{R}^r)\oplus\mathbb{R}$
I_{14B}	$\mathbb{R}\ltimes\mathbb{R}^r$ (type II)	$y,\ -x,\ -\int^x\eta_j(x')\mathrm{d}x',\ 1$	$\mathrm{d}x\wedge\mathrm{d}y$	$\overline{(\mathbb{R}\ltimes\mathbb{R}^r)}$
I_{16}	$C_{-1}^r\simeq\mathfrak{h}_2\ltimes\mathbb{R}^{r+1}$	$y,\ -x,\ xy,\ -\dfrac{x^{j+1}}{j+1},\ 1$	$\mathrm{d}x\wedge\mathrm{d}y$	$\overline{\mathfrak{h}_2\ltimes\mathbb{R}^{r+1}}$

Applications of Lie–Hamilton systems on $\mathbb{R}^2$

LH algebra	#	LH systems
$\mathfrak{sl}(2)$	P_2	Milne–Pinney and SOKSE with $c > 0$
		Complex Riccati equation
$\mathfrak{sl}(2)$	I_4	Milne–Pinney and SOKSE with $c < 0$
		Split-complex Riccati equation
		Planar diffusion Riccati system for $c_0 = 1$
$\mathfrak{sl}(2)$	I_5	Milne–Pinney and SOKSE with $c = 0$
		Dual-Study Riccati equation
		Harmonic oscillator
		Planar diffusion Riccati system for $c_0 = 0$
$\mathfrak{h}_6 \simeq \overline{\mathfrak{sl}(2) \ltimes \mathbb{R}^2}$	P_5	Dissipative harmonic oscillator
		SORE in Hamiltonian form
$\mathfrak{h}_2 \simeq \mathbb{R} \ltimes \mathbb{R}$	$I_{14A}^{r=1}$	Complex Bernoulli equation
		Generalized Buchdahl equations
		Lotka–Volterra systems

The coalgebra method

If X is a Lie–Hamilton system with a Lie–Hamiltonian structure (N, Λ, h), then the space $(S_{\mathfrak{g}}, \cdot, \{\cdot,\cdot\}_{S_{\mathfrak{g}}}, \Delta)$, with $\mathfrak{g} \simeq (\mathfrak{h}, \{\cdot,\cdot\}_{\Lambda})$, is a **Poisson coalgebra** with a **coproduct** $\Delta(v) = v \otimes 1 + 1 \otimes v, \quad \forall v \in \mathfrak{g} \subset S_{\mathfrak{g}}$.

Theorem

Given a **Casimir element** C *of* $(S_{\mathfrak{g}}, \cdot, \{,\}_{S_{\mathfrak{g}}})$, *where* $\mathfrak{g} \simeq \mathfrak{h}$, *then*
(i) The functions defined as

$$F^{(k)} = D^{(k)}(\Delta^{(k)}(C)), \qquad k = 2, \ldots, m,$$

are t-independent constants of motion for $\widetilde{X}$ *to* N^m. *Furthermore, they form a set of* $(m-1)$ *functionally independent functions in involution.*
(ii) The functions given by

$$F_{ij}^{(k)} = S_{ij}(F^{(k)}), \quad 1 \leq i < j \leq k, \quad k = 2, \ldots, m,$$

S_{ij} *is the permutation of variables* $x_{(i)} \leftrightarrow x_{(j)}$, *are t-independent constants of motion for* $\widetilde{X}$ *to* N^m.

The Winternitz–Smorodinsky system

Consider the n-**dimensional Smorodinsky–Winternitz system** whose Hamiltonian reads

$$h = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\omega^2(t)\sum_{i=1}^{n} x_i^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{b_i}{x_i^2},$$

where the b_i's are n real constants. The corresponding Hamilton's equations read

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = p_i, \\ \dfrac{\mathrm{d}p_i}{\mathrm{d}t} = -\omega^2(t)x_i + \dfrac{b_i}{x_i^3}, \end{cases} \qquad i = 1, \ldots, n.$$

This system describes the integral curves of the t-dependent vector field on $\mathrm{T}^*\mathbb{R}_0^n = \mathrm{T}^*\mathbb{R}^n - \{0\}$ given by $X = X_3 + \omega^2(t)X_1$, where the vector fields

$$X_1 = -\sum_{i=1}^{n} x_i\frac{\partial}{\partial p_i}, \qquad X_2 = \frac{1}{2}\sum_{i=1}^{n}\left(p_i\frac{\partial}{\partial p_i} - x_i\frac{\partial}{\partial x_i}\right),$$

$$X_3 = \sum_{i=1}^{n}\left(p_i\frac{\partial}{\partial x_i} + \frac{b_i}{x_i^3}\frac{\partial}{\partial p_i}\right),$$

which close a VG *isomorphic to* $\mathfrak{sl}(2,\mathbb{R})$.

The Winternitz–Smorodinsky system

The space $\mathrm{T}^*\mathbb{R}_0^n$ admits a natural Poisson bivector $\Lambda = \sum_{i=1}^n \partial/\partial x_i \wedge \partial/\partial p_i$. with respect to which the Hamiltonian functions are

$$h_1 = \frac{1}{2}\sum_{i=1}^n x_i^2, \qquad h_2 = -\frac{1}{2}\sum_{i=1}^n x_i p_i, \qquad h_3 = \frac{1}{2}\sum_{i=1}^n \left(p_i^2 + \frac{b_i}{x_i^2}\right)$$

closing a Poisson Lie algebra *isomorphic to* $\mathfrak{sl}(2,\mathbb{R})$.

We derive an explicit superposition rule for $n=1$. We need two first integrals for $\widetilde{X}_\alpha$ on $\{x_{(1)}, p_{(1)}, x_{(2)}, p_{(2)}, x_{(3)}, p_{(3)}\}$ of $\mathrm{T}^*\mathbb{R}_0^3$. Given the Casimir $C = h_1 h_3 - h_2^2$ of $\mathfrak{sl}(2,\mathbb{R})$

$$F^{(2)} = D^{(2)}(\Delta(C)) = \frac{1}{4}(x_1 p_2 - x_2 p_1)^2 + \frac{b(x_1^2 + x_2^2)^2}{4x_1^2 x_2^2},$$

$$F^{(3)} = D^{(3)}(\Delta^3(C)) = \frac{1}{4}\sum_{1\le i<j}^{3}\left[(x_i p_j - x_j p_i)^2 + \frac{b(x_i^2+x_j^2)^2}{x_i^2 x_j^2}\right] - \frac{3}{4}b,$$

$$F_{13}^{(2)} = S_{13}(F^{(2)}),$$

$$F_{23}^{(2)} = S_{23}(F^{(2)}), \quad F^{(3)} = F^{(2)} + F_{13}^{(2)} + F_{23}^{(2)} - 3b/4,$$

The Winternitz–Smorodinsky system

We choose $F^{(2)}$ and $F_{23}^{(2)}$ as the two functionally independent constants of motion and we shall use $F_{13}^{(2)}$ in order to simplify the result.
Indeed, we set

$$F^{(2)} = \frac{k_1}{4} + \frac{b}{2}, \quad F_{23}^{(2)} = \frac{k_2}{4} + \frac{b}{2}, \quad F_{13}^{(2)} = \frac{k_3}{4} + \frac{b}{2},$$

$$\begin{aligned} x_1 &= x_1(x_2, p_2, x_3, p_3, k_1, k_2) = x_1(x_2, x_3, k_1, k_2, k_3) \\ &= \left\{ \mu_1 x_2^2 + \mu_2 x_3^2 \pm \sqrt{\mu \left[k_3 x_2^2 x_3^2 - b(x_2^4 + x_3^4) \right]} \right\}^{1/2}, \end{aligned}$$

where the constants μ_1, μ_2, μ are defined in terms of k_1, k_2, k_3 and b as

$$\mu_1 = \frac{2bk_1 - k_2 k_3}{4b^2 - k_3^2}, \quad \mu_2 = \frac{2bk_2 - k_1 k_3}{4b^2 - k_3^2},$$

$$\mu = \frac{4\left[4b^3 + k_1 k_2 k_3 - b(k_1^2 + k_2^2 + k_3^2)\right]}{(4b^2 - k_3^2)^2}.$$

$$p_1 = p_1(x_1, x_2, p_2, x_3, p_3, k_1) = \frac{p_2 x_1^2 x_2 \pm \sqrt{k_1 x_1^2 x_2^2 - b(x_1^4 + x_2^4)}}{x_1 x_2^2}.$$

Dirac–Lie systems

Theorem

(Lie–Hamilton no-go Theorem) *If X is a Lie system on an odd-dimensional manifold N satisfying that $\mathcal{D}^X_{x_0} = T_{x_0}N$ for a point x_0 in N, then X is not a Lie–Hamilton system on N.*

The third-order Kummer–Schwarz equations (TOKSE)

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = a, \quad \frac{da}{dt} = \frac{3}{2}\frac{a^2}{v} - 2c_0(x)v^3 + 2b_1(t)v,$$

are Hamiltonian w.r.t. a presymplectic manifold $(\mathcal{O}_2, \omega_{3KS} = \frac{dv\wedge da}{v^3})$ and with Hamiltonian functions

$$h_1 = -\frac{2}{v}, \qquad h_2 = -\frac{a}{v^2}, \qquad h_3 = -\frac{a^2}{2v^3} - 2c_0 v,$$
$$\{h_1, h_3\} = 2h_2, \quad \{h_1, h_2\} = h_1, \quad \{h_2, h_3\} = h_3,$$

Definition

A **Dirac–Lie Hamiltonian structure** is a triple (N, L, h), where (N, L) stands for a Dirac manifold, $h_t : N \to \mathbb{R}$ st. $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_L)$ is a finite-dimensional real Lie algebra. X admits a Dirac–Lie Hamiltonian (N, L, h) if $X_t + dh_t \in \Gamma(L)$.

Construction of Bi-Dirac–Lie systems

Consider a Dirac–Lie system (N, L^ω, X) and a t-independent Lie symmetry Z of X. Then, $\omega_Z = \mathcal{L}_Z\omega$ satisfies $d\omega_Z = d\mathcal{L}_Z\omega = \mathcal{L}_Z d\omega = 0$, so (N, ω_Z) is another presymplectic manifold.

Example

TOKSE with $c_0 = 0$, possess a Lie symmetry $Z = x^2\partial/\partial x + 2vx\frac{\partial}{\partial v} + 2(ax + v^2)\frac{\partial}{\partial a}$,

$$\omega_Z \equiv \mathcal{L}_Z\omega_{3KS} = -\frac{2}{v^3}(xdv \wedge da + vda \wedge dx + adx \wedge dv).$$

Moreover,

$$\iota_{Y_1}\omega_Z = -d(Zh_1) = -d\left(\frac{4x}{v}\right), \quad \iota_{Y_2}\omega_Z = -d(Zh_2) = d\left(2 - \frac{2ax}{v^2}\right),$$
$$\iota_{Y_3}\omega_Z = -d(Zh_3) = d\left(\frac{2a}{v} - \frac{a^2x}{v^3}\right).$$

$$\{Zh_1, Zh_2\}_{L^{\omega_Z}} = Zh_1, \quad \{Zh_2, Zh_3\}_{L^{\omega_Z}} = Zh_3, \quad \{Zh_1, Zh_3\}_{L^{\omega_Z}} = 2Zh_2$$

If $(\mathcal{O}_2, L^\omega, h)$ is a Lie–Hamiltonian for X, then $(\mathcal{O}_2, L^{\omega_Z}, Zh)$ is a Dirac–Lie Hamiltonian.

Jacobi–Lie systems

Definition

A **Jacobi–Lie system** (N, Λ, R, X) consists of the Jacobi manifold (N, Λ, R) and a Lie system X admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to (N, Λ, R).

We consider the **continuous Heisenberg group** which describes the space of matrices

$$\mathbb{H} = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| x, y, z \in \mathbb{R} \right\},$$

The bivector $\Lambda_{\mathbb{H}}$ is given by $\Lambda_{\mathbb{H}} \equiv -y \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ and $R_{\mathbb{H}} \equiv \partial/\partial z$. Then,

$$X_1^L = [\Lambda_{\mathbb{H}}, -y]_{SN} - yR_{\mathbb{H}}, \qquad X_2^L = [\Lambda_{\mathbb{H}}, x]_{SN} + xR_{\mathbb{H}}, \qquad X_3^L = [\Lambda_{\mathbb{H}}, 1]_{SN} + R_{\mathbb{H}}.$$

where the left-invariant vector fields on $\mathbb{H}$ are

$$X_1^L = \frac{\partial}{\partial x}, \qquad X_2^L = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad X_3^L = \frac{\partial}{\partial z}.$$

Classification table of Jacobi–Lie systems

#	Lie algebra	Basis of vector fields X_i	Jacobi
P_1	$A_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, \alpha(x\partial_x + y\partial_y) + y\partial_x - x\partial_y, \quad \alpha \geq 0$	$(\alpha = 0)$ Pois.
P_2	$\mathfrak{sl}(2)$	$\partial_x, x\partial_x + y\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y$	Poisson
P_3	$\mathfrak{so}(3)$	$y\partial_x - x\partial_y, (1 + x^2 - y^2)\partial_x + 2xy\partial_y,$ $2xy\partial_x + (1 + y^2 - x^2)\partial_y$	Poisson
P_4	$\mathbb{R}^2 \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y$	No
P_5	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_y$	Poisson
P_6	$\mathfrak{gl}(2) \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y$	No
P_7	$\mathfrak{so}(3,1)$	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y,$ $2xy\partial_x + (y^2 - x^2)\partial_y$	No
P_8	$\mathfrak{sl}(3)$	$\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y$	No
I_1	$\mathbb{R}$	∂_x	$(0, \partial_x)$, Pois.
I_2	$\mathfrak{h}_2$	$\partial_x, x\partial_x$	$(0, \partial_x)$

Conclusions: Lie systems

- Our work on SORE has avoided their mapping into $\mathfrak{sl}(3,\mathbb{R})$-Lie systems. By means of a Legendre transformation, SORE reduce to solving Lie systems in $\mathfrak{sl}(2,\mathbb{R})$.
- We have detailed numerous applications of Lie systems. We have expanded their applications to biological and medical models: virus spread models and Lotka Volterra. Others are systems with nonlinearities, Bernoulli equations, etc. Also PDE-Riccati equations.
- We have found that only 12 out of the 28 initial classes in the GKO, are Lie–Hamilton systems. A lot of examples have been displayed, each belonging to a different class in our classification.
- We have proven the efficiency of the coalgebra method for obtaining superposition rules and their application to different systems.
- For structure-Lie systems, we have proposed Dirac–Lie systems and Jacobi–Lie systems. We have provided a classification of planar Lie algebras with respect to a Jacobi structure.

Publications on Lie systems

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PART II: LIE SYMMETRIES

1. Lie symmetries for Lie systems: *Lie algebras of Lie symmetries. Applications to ODE-Lie systems*
2. The nonclassical Lie symmetry method: *algorithmic procedure and reduction of equations and their associated Lax pairs, to lower dimensions*
    - Application to hierarchies of PDEs: Symmetries and reduction of the Camassa–Holm and Qiao hierarchies in $2+1$ dimensions. Symmetries and reductions of their corresponding Lax pairs.

Lie symmetry for Lie systems

Given a Lie system $X = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha$, we study its Lie symmetries of the form

$$Y = f_0(t)\frac{\partial}{\partial t} + \sum_{\alpha=1}^{r} f_\alpha(t) X_\alpha,$$

where $f_0, \ldots, f_r$ are certain t-dependent functions. We denote by $\mathcal{S}_X^V$ **the space of Lie symmetries**. In order for Y to be a symmetry of X, it needs to be fulfilled

$$\frac{\mathrm{d}f_0}{\mathrm{d}t} = b_0(t), \quad \frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = f_0 \frac{\mathrm{d}b_\alpha}{\mathrm{d}t}(t) + b_\alpha(t) b_0(t) + \sum_{\beta,\gamma=1}^{r} b_\beta(t) f_\gamma c_{\gamma\beta\alpha},$$

is satisfied for a certain t-dependent function b_0 and $\alpha = 1, \ldots, r$.

Definition

We call the **symmetry system** of the Lie system with respect to its VG.

$$\Gamma_X^V = b_0(t)\frac{\partial}{\partial f_0} + \sum_{\alpha=1}^{r}\left(f_0 \frac{\mathrm{d}b_\alpha}{\mathrm{d}t}(t) + b_0(t) b_\alpha(t) + \sum_{\gamma,\beta=1}^{r} b_\beta(t) f_\gamma c_{\gamma\beta\alpha}\right)\frac{\partial}{\partial f_\alpha}.$$

Lie symmetry for Lie systems: main theorem

Theorem

The system Γ_X^V is a Lie system possessing a VG

$$(A_1 \oplus_S A_2) \oplus_S V_L \simeq (\mathbb{R}^{r+1} \oplus_S \mathbb{R}^r) \oplus_S V/Z(V),$$

where

$$A_1 = \langle Z_0, \ldots, Z_r \rangle \simeq \mathbb{R}^{r+1}, \qquad A_2 = \langle W_1, \ldots, W_r \rangle \simeq \mathbb{R}^r,$$
$$V_L = \langle Y_1, \ldots, Y_r \rangle \simeq V/Z(V),$$

with

$$Y_\alpha = \sum_{\beta,\gamma=1}^{r} f_\beta c_{\beta\alpha\gamma} \frac{\partial}{\partial f_\gamma}, \qquad W_\alpha = f_0 \frac{\partial}{\partial f_\alpha}, \qquad Z_0 = \frac{\partial}{\partial f_0}, \qquad Z_\alpha = \frac{\partial}{\partial f_\alpha},$$

with $\alpha = 1, \ldots, r$, we write $A \oplus_S B$ for the semi-direct sum of the ideal A of $A + B$ with B, and $Z(V)$ is the center of the Lie algebra V.

Application to HODEs and PDEs

This has been illustrated by the application of this procedure to

- $\mathfrak{sl}(2,\mathbb{R})$-Lie systems
    1. A particular first-order Riccati equation
    2. The Cayley–Klein Riccati equation
    3. The generalized Darboux–Brioschi–Halphen systems
    4. Second–order Kummer–Schwarz equations
- Aff($\mathbb{R}$)-Lie systems
    1. Buchdahl equation
- PDE-Lie systems
    1. Partial differential Riccati equations

The CHH$(2+1)$

The **negative Camassa–Holm hierarchy in** $2+1$ (NCHH$2+1$) reads

$$\begin{aligned} &u_y = (v_{[n]})_x - (v_{[n]})_{xxx}, \\ &(v_{[j-1]})_x - (v_{[j-1]})_{xxx} = u_x v_{[j]} + 2u(v_{[j]})_x, \quad j = 2, \dots, n \\ &u_t = u_x v_{[1]} + 2u(v_{[1]})_x. \end{aligned}$$

Reduction $X = T$ and $n = 1$ recovers the *Camassa–Holm equation*.

The *Lax pair associated with the NCHH*$(2+1)$ is a nonisospectral Lax pair

$$\psi_{xx} - \left(\frac{1}{4} - \frac{\lambda}{2}u\right)\psi = 0,$$

$$\psi_y - \lambda^n \psi_t + \hat{\mathcal{A}}\psi_x - \frac{\hat{\mathcal{A}}_x}{2}\psi = 0,$$

where $\hat{\mathcal{A}} = \sum_{i=1}^{n} \lambda^{(n-j+1)} v_{[j]}$ and the *nonisospectrality condition*.

$$\lambda_y - \lambda^n \lambda_t = 0,$$

Nonclassical symmetries

The nonclassical symmetries impose the following **invariant surface conditions**

$$\begin{aligned}
&\eta_u = \xi_1 u_x + \xi_2 u_y + \xi_3 u_t,\\
&\eta_{v_{[j]}} = \xi_1 (v_{[j]})_x + \xi_2 (v_{[j]})_y + \xi_3 (v_{[j]})_t, \quad j = 1, \ldots, n.\\
&\eta_\lambda = \xi_2 \lambda_y + \xi_3 \lambda_t,\\
&\eta_\psi = \xi_1 \psi_x + \xi_2 \psi_y + \xi_3 \psi_t.
\end{aligned}$$

We present a classification attending to different choices of the infinitesimal generator

Values of infinitesimal generators			
Case A.	$\xi_3 = 1$	any	any
Case B.	$\xi_3 = 0$	$\xi_2 = 1$	any
Case C.	$\xi_3 = 0$	$\xi_2 = 0$	$\xi_1 = 1$

Possible values of the t-dependent functions			
Case I.	$A_1(t) \neq 0$	$B_1(t) = 0$	$C_1(t) = 0$
Case II	$A_1(t) = 0$	$B_1(t) \neq 0$	$C_1(t) = 0$
Case III.	$A_1(t) = 0$	$B_1(t) = 0$	$C_1(t) \neq 0$

Case A. Nonclassical symmetries

Under these conditions, we obtain the following *nonclassical symmetries*

$$\xi_1 = S_1/S_3, \quad \xi_2 = S_2/S_3, \quad \xi_3 = 1$$

$$\eta_\lambda = \frac{1}{S_3}\left(\frac{a_3 - a_2}{n}\right)\lambda, \quad \eta_\psi = \frac{1}{S_3}\left(\frac{1}{2}\frac{\partial S_1}{\partial x} + a_0\right)\psi.$$

$$\eta_u = \frac{1}{S_3}\left(\frac{1}{2}\frac{\partial S_1}{\partial x} + a_0\right)\psi,$$

$$\eta_{v_{[1]}} = \frac{1}{S_3}\left[v_{[1]}\left(\frac{\partial S_1}{\partial x} - a_3\right) - \frac{\partial S_1}{\partial t}\right],$$

$$\eta_{v_{[j]}} = \frac{1}{S_3}\left(\frac{\partial S_1}{\partial x} - a_2\frac{j-1}{n} - a_3\frac{n-j+1}{n}\right)v_{[j]}, \quad j = 2, \ldots, n.$$

where

$$S_1 = S_1(x,t) = A_1(t) + B_1(t)e^x + C_1(t)e^{-x},$$
$$S_2 = S_2(y) = a_2 y + b_2, \quad S_3 = S_3(t) = a_3 t + b_3.$$

with $A_1(t), A_2(t), A_3(t)$ arbitrary functions of t. Furthermore, a_0, a_2, b_2, a_3, b_3 are arbitrary constants, such that a_3 and b_3 cannot be simultaneously 0.

Reductions for $\xi_3 \neq 0$

Case II yields the same reduced spectral problems as the obtained for I. Case III is easy to prove that it is equivalent to II owing to the invariance of the Lax pair and equations under the transformation $x \to -x$, $y \to -y$ and $t \to -t$.

We introduce the notation for the reduced variables as

$$x, y, t \to x_1, x_2$$

and the reduced fields and reduced eigenfunction

$$u(x,y,t) \to U(x_1,x_2), \quad v_{[1]}(x,y,t) \to V_{[1]}(x_1,x_2), \quad v_{[j]}(x,y,t) \to V_{[j]}(x_1,x_2),$$
$$\lambda(y,t) \to \Lambda(x_2), \quad \psi(x,y,t) \to \Phi(x_1,x_2).$$

We obtain 5 different nontrivial reductions.

Values of the constants			
1.	$a_2 = 0$	$a_3 = 0$	$b_2 = 0$
2.	$a_2 = 0$	$a_3 = 0$	$b_2 \neq 0$
3.	$a_2 = 0$	$a_3 \neq 0$	$b_2 = 0$
4.	$a_2 = 0$	$a_3 \neq 0$	$b_2 \neq 0$
5.	$a_2 \neq 0$	any	any

I.1. $B_1(t) = C_1(t) = 0, A_1(t) \neq 0, a_2 = 0, a_3 = 0, b_2 = 0$

- Reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial V_{[n]}}{\partial x_1} + \frac{\partial U}{\partial x_2} = 0,$$
$$\frac{2U\partial V_{[1]}}{\partial x_1} + V_{[1]}\frac{\partial U}{\partial x_1} = 0,$$
$$\frac{2U\partial V_{[j+1]}}{\partial x_1} + V_{[j+1]}\frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} - \frac{\partial V_{[j]}}{\partial x_1} = 0.$$

it is the *positive Camassa-Holm hierarchy*, for $n = 1$ is a *modified Dym equation*.

- Reduced Lax pair

$$\Phi_{x_1 x_1} - \left(\frac{1}{4} - \frac{\lambda_0}{2}U\right)\Phi = 0,$$
$$\Phi_{x_2} + \hat{\mathcal{B}}\Phi_{x_1} - \frac{\hat{\mathcal{B}}}{2}\Phi = 0,$$

with $\hat{\mathcal{B}} = \sum_{j=1}^{n} \lambda_0^{n-j+1} V_{[j]}(x_1, x_2)$. It is an *isospectral reduction*.

Case B. Nonclassical symmetry

The second type of symmetry, $\xi_3 = 0, \xi_2 = 1$ leads us to

$$\begin{aligned}
\xi_1 &= S_1/S_2, \quad \xi_2 = 1, \quad \xi_3 = 0,\\
\eta_\lambda &= \frac{1}{S_2}\left(-\frac{a_2}{n}\right)\lambda,\\
\eta_u &= \frac{1}{S_2}\left(-2\frac{\partial S_1}{\partial x} + \frac{a_2}{n}\right)u,\\
\eta_{v_{[1]}} &= \frac{1}{S_2}\left(\frac{\partial S_1}{\partial x}v_{[1]} - \frac{\partial S_1}{\partial t}\right),\\
\eta_{v_{[j]}} &= \frac{1}{S_2}\left(\frac{\partial S_1}{\partial x} - a_2\frac{j-1}{n}\right)v_{[j]}, \quad j = 2, \ldots, n.\\
\eta_\psi &= \frac{1}{S_2}\left(-2\frac{\partial S_1}{\partial x} + \frac{a_2}{n}\right)\psi,
\end{aligned}$$

Evidently, a_2 and b_2 cannot be 0 at the same time.

Case B. Nonclassical symmetry

One of the reduced variables is t. This means that the integrals that involve S_1 can be performed without any restrictions for the functions $A_1(t)$, $B_1(t)$, $C_1(t)$.

Values of the functions		
1.	$a_2 = 0$	$E = \sqrt{A_1^2 - 4B_1C_1} = 0$
2.	$a_2 = 0$	$E = \sqrt{A_1^2 - 4B_1C_1} \neq 0$
3.	$a_2 \neq 0$	$E = \sqrt{A_1^2 - 4B_1C_1} = 0$
4.	$a_2 \neq 0$	$E = \sqrt{A_1^2 - 4B_1C_1} \neq 0$

Reduction 2. $a_2 = 0,\ E = \sqrt{A_1^2 - 4B_1C_1} \neq 0$

- Reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial V_{[n]}}{\partial x_1} - \frac{\partial U}{\partial x_1} = 0,$$
$$2U\frac{\partial V_{[1]}}{\partial x_1} + V_{[1]}\frac{\partial U}{\partial x_1} - \frac{\partial U}{\partial x_2} = 0,$$
$$2U\frac{\partial V_{[j+1]}}{\partial x_1} + V_{[j+1]}\frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} - \frac{\partial V_{[j]}}{\partial x_1} = 0.$$

- Reduced Lax pair

$$\Phi_{x_1x_1} + \left(\frac{\lambda_0}{2}U - \frac{1}{4}\right)\Phi = 0,$$
$$\lambda_0^n\Phi_{x_2} = (\hat{\mathcal{B}} - 1)\Phi_{x_1} - \frac{\hat{\mathcal{B}}_{x_1}}{2}\Phi,$$

with $\hat{\mathcal{B}} = \sum_{j=1}^{n} \Lambda(x_2)^{n-j+1} V_{[j]}(x_1, x_2)$.

which is the celebrated *negative Camassa–Holm hierarchy* and it is an *isospectral* reduction.

The Qiao hierarchy

Another important hierarchy is the **Qiao hierarchy** (mCHH$(2+1)$).

$$
\begin{aligned}
&u_y = -(u\omega_{[1]})_x,\\
&(v_{[j]})_{xx} - v_{[j]} = -u\omega_{[j+1]}, \qquad j = 1, \dots, n-1.\\
&(\omega_{[j]})_x = u(v_{[j]})_x,\\
&u_t = ((v_{[n]})_{xx} - v_{[n]})_x,
\end{aligned}
$$

Reduction $y = 0$ and $n = 1$ recovers the *Qiao equation*.

This hierarchy is associated with a two component Lax pair

$$
\begin{pmatrix} \phi \\ \psi \end{pmatrix}_t = \lambda^n \begin{pmatrix} \phi \\ \psi \end{pmatrix}_y + \lambda p \begin{pmatrix} \phi \\ \psi \end{pmatrix}_x + \frac{i\sqrt{\lambda}}{2} \begin{bmatrix} 0 & q_x - q \\ q_x + q & 0 \end{bmatrix}_x \begin{pmatrix} \phi \\ \psi \end{pmatrix},
$$

$$
\begin{pmatrix} \phi \\ \psi \end{pmatrix}_x = \frac{1}{2} \begin{bmatrix} -1 & i\sqrt{\lambda}u \\ i\sqrt{\lambda}u & 1 \end{bmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}.
$$

where $p = \sum_{j=1}^{n} \lambda^{n-j}(y,t)\omega_{[j]}(x,y,t)$ and $q = \sum_{j=1}^{n} \lambda^{n-j}(y,t)v_{[j]}(x,y,t)$.

From the compatibility condition, we obtain the *nonisospectrality condition*

$$
\lambda_t - \lambda^n \lambda_y = 0
$$

.

Classical Lie symmetries

The classical symmetries for the mCHH$(2+1)$ result in

$$\xi_1 = A_1(t), \quad \xi_2 = a_2 y + b_2, \quad \xi_3 = a_3 t + b_3,$$
$$\eta_\lambda(y,t,\lambda) = \frac{a_2 - a_3}{n}\lambda,$$
$$\eta_u(x,y,t,u) = \frac{a_3 - a_2}{2n}u,$$
$$\eta_{\omega_{[j]}}(x,y,t,\omega_{[j]}) = \delta^{(j,1)}\frac{dA_1(y)}{dy} - \frac{(n-j+1)a_2 + (j-1)a_3}{n}\omega_{[j]},$$
$$\eta_{v_{[j]}}(x,y,t,v_{[j]}) = \delta^{(j,n)}A_n(y,t) - \frac{(2(n-j)+1)a_2 + (2j-1)a_3}{2n}v_{[j]}, \quad j = 1,\dots,n$$
$$\eta_\phi(x,y,t,\lambda,\psi,\phi) = \gamma(y,t,\lambda)\phi,$$
$$\eta_\psi(x,y,t,\lambda,\psi,\phi) = \gamma(y,t,\lambda)\psi,$$

where the function γ satisfies the following equation

$$\frac{\partial\gamma(y,t,\lambda)}{\partial t} = \lambda^n\frac{\partial\gamma(y,t,\lambda)}{\partial t},$$

The functions are $A_1(y)$, $A_n(y,t)$ and constants a_2, a_3, b_2, b_3 are arbitrary and $\delta^{(j,1)}$ and $\delta^{(j,n)}$ are Kronecker deltas.

Classical Lie symmetries

If we introduce the next notation for the reduced variables and vector fields

$$
\begin{aligned}
&x, y, t, \to x_1, x_2, \quad U(x,y,t) \to U(z_1, z_2),\\
&\omega_{[1]}(x,y,t) \to \Omega_{[1]}(x_1,x_2), \quad \omega_{[j]}(x,y,t) \to \Omega_{[j]}(x_1,x_2), \quad j = 2, \dots, n,\\
&v_{[j]}(x,y,t) \to V_{[j]}(x_1,x_2), \quad v_{[n]}(x,y,t) \to V_{[n]}(x_1,x_2), \quad j = 1, \dots, n-1,\\
&\phi(x,y,t) \to \Phi(x_1,x_2), \quad \psi(x,y,t) \to \Psi(x_1,x_2), \quad \lambda \to \Lambda(x_2),\\
&p(x,y,t) \to P(x_1,x_2), \quad q(x,y,t) \to Q(x_1,x_2).
\end{aligned}
$$

Case I: $a_2 \neq 0, b_2 = 0$			Case II: $a_2 = 0, b_2 \neq 0$		Case III: $a_2 = 0, b_2 = 0$	
1.	$a_3 \neq 0$	$b_3 = 0$	$a_3 \neq 0$	$b_3 = 0$	$a_3 \neq 0$	$b_3 = 0$
2.	$a_3 = 0$	$b_3 \neq 0$	$a_3 = 0$	$b_3 \neq 0$	$a_3 = 0$	$b_3 \neq 0$
3.	$a_3 = 0$	$b_3 = 0$	$a_3 = 0$	$b_3 = 0$		

Table: Classification of reductions for mCHH$2+1$

Reduction I.1. $a_2 \neq 0, b_2 = 0, a_3 \neq 0, b_3 = 0$

- Reduced hierarchy

$$\left((V_{[n]})_{x_1 x_1} - V_{[n]}\right)_{x_1} - U_{x_2} = 0,$$
$$(V_{[j]})_{x_1 x_1} - V_{[j]} + U\Omega_{[j+1]} = 0, \qquad j = 1, \dots, n-1,$$
$$\left(\Omega_{[1]} U\right)_{x_1} + \frac{r-1}{2n} U - r x_2 U_{x_2} = 0,$$
$$(\Omega_{[j]})_{x_1} = U (V_{[j]})_{x_1}.$$

where $r = a_3/a_2$.

- The reduced spectral problem is *nonisospectral*, $\frac{\Lambda(x_2)}{dx_2} = \frac{1-r}{n} \frac{\Lambda(x_2)^{n+1}}{1+rx_2\Lambda(x_2)^n}$

$$(1 + r x_2 \Lambda^n)\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_2} = \Lambda P \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{x_1} + \frac{i\sqrt{\Lambda}}{2}\begin{bmatrix} 0 & Q_{x_1} - Q \\ Q_{x_1} + Q & 0 \end{bmatrix}_{x_1} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

and

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} = \frac{1}{2}\begin{bmatrix} -1 & i\sqrt{\Lambda} U \\ i\sqrt{\Lambda} U & 1 \end{bmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

Conclusions

- We have calculated Lie symmetries of ordinary and PDE-Lie systems. The resulting Lie symmetries form a Lie algebra that is related to the VG of the Lie system. An important theorem relating the Lie algebra of symmetries and the VG has been obtained and illustrated by means of several examples.
- We have calculated point Lie symmetries for higher-order hierarchies of PDEs and their corresponding Lax pairs.

  - In the case of the CHH$(2+1)$ and its non-isospectral Lax pair, we have performed the nonclassical approach. Each of possibility comes from the choice of the 5 arbitrary constants and 3 arbitrary time dependent functions that appear in the symmetries. Out of the total 9 nontrivial cases, 5 have a nonisopectral Lax pair, 2 of them are the positive and negative Camassa–Holm hierarchies.

  - We have dealt with the mCHH$(2+1)$ and its two component Lax pair. We have calculated its classical Lie symmetries. Such symmetries depend on 4 constants and 2 arbitrary functions. There result 8 different reductions, of which a few are nonisospectral even in $1+1$.

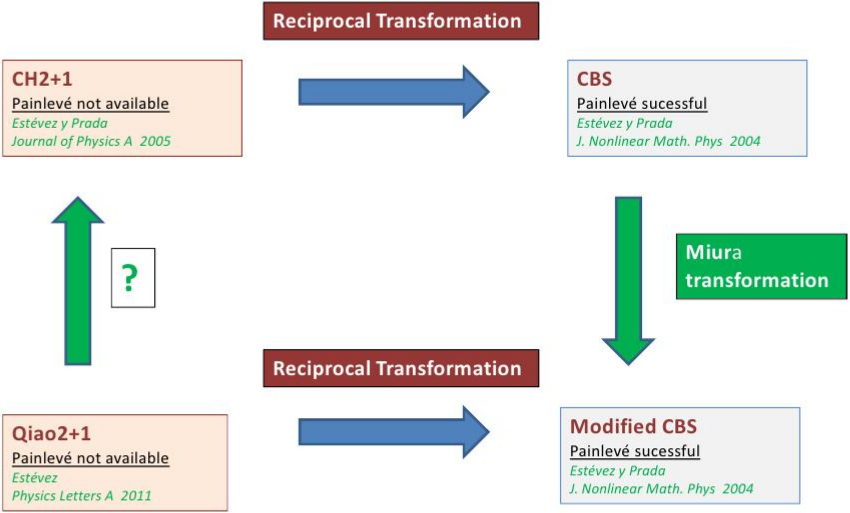
Publications on Lie symmetries

- Similarity reductions arising from nonisospectral Camassa Holm hierarchy in 2+1 dimensions,
  P.G. Estévez, J.D. Lejarreta, C. Sardón,
  J. Nonl. Math. Phys. **18**, 9–28 (2011).
- Integrable 1+1 dimensional hierarchies arising from reduction of a non-isospectral problem in 2+1 dimensions,
  P.G. Estévez, J.D. Lejarreta, C. Sardón,
  Appl. Math. Comput. **224**, 311–324 (2013).
- Lie symmetries for Lie systems: applications to systems of ODEs and PDEs,
  P.G. Estévez, F.J. Herranz, J. de Lucas, C. Sardón,
  submitted to *Appl. Math. Comput.*
  arXiv:1404.2740

PART III: MIURA-RECIPROCAL TRANSFORMATIONS

1. The CHH$(2+1)$ versus the CBS equation
2. The mCHH$(2+1)$ versus the mCBS equation
3. Relations between mCHH$(2+1)$ and CHH$(2+1)$ and their corresponding Lax pairs

Motivation



Reciprocal Transformation
CH2+1
Painlevé not available
Estévez y Prada
Journal of Physics A 2005
CBS
Painlevé sucessful
Estévez y Prada
J. Nonlinear Math. Phys 2004
?
Miura transformation
Reciprocal Transformation
Qiao2+1
Painlevé not available
Estévez
Physics Letters A 2011
Modified CBS
Painlevé sucessful
Estévez y Prada
J. Nonlinear Math. Phys 2004

The CHH$(2+1)$ versus the CBS equation

We propose the change (X, Y, T) in CHH$(2+1)$ to (z_0, z_1, z_{n+1}), where $P^2 = U$

$$dz_0 = PdX - \frac{1}{2}P\Omega_{[1]}dY + \Delta dT, \quad dz_1 = dY, \quad dz_{n+1} = dT.$$

We introduce the old independent variable as a dependent field $X = X(z_0, z_1, z_{n+1})$ such that

$$dX = X_{z_0}dz_0 + X_{z_1}dz_1 + X_{z_{n+1}}dz_{n+1}, \quad dX = \frac{1}{P}dz_0 + \frac{1}{2}\Omega_{[1]}dz_1 - \frac{\Delta}{P}dz_{n+1}.$$

By direct comparison, $\quad X_{z_0} = \frac{1}{P}, \quad X_{z_1} = \frac{1}{2}\Omega_{[1]}, \quad X_{z_{n+1}} = -\frac{\Delta}{P}.$

We add variables $z_2, \dots, z_n$ such that $\Omega_{[i]} = 2X_{z_i}, \forall i = 2, \dots, n$ and $X_{z_i} = \frac{\partial X}{\partial z_i}$

$$-\left(\frac{X_{z_{i+1}}}{X_{z_0}}\right)_{z_0} = \left(\frac{X_{z_0z_0z_0}}{X_{z_0}} - \frac{3}{2}\frac{X^2_{z_0z_0}}{X^2_{z_0}} - \frac{1}{2}X^2_{z_0}\right)_{z_i}, \forall i = 1, \dots, n.$$

which can be rewritten as can be considered as n copies of the Calogero-Bogoyanlevskii-Schiff **CBS equation** if we call $-4M_{z_i}$ to the left-hand term and $4M_{z_0}$ to the right-hand term.

$$M_{z_0,z_{i+1}} + M_{z_0z_0z_0z_i} + 4M_{z_i}M_{z_0z_0} + 8M_{z_0}M_{z_iz_0} = 0.$$

The mCHH$(2+1)$ versus the mCBS equation

We propose the change (x, y, t) to (z_0, z_1, z_{n+1})

$$dz_0 = udx - u\omega_{[1]}dy + \delta dt, \quad dz_1 = dy, \quad dz_{n+1} = dt.$$

We introduce the old independent variable as a dependent field $x = x(z_0, z_1, z_n)$ such that

$$dx = x_0 dz_0 + x_1 dz_1 + x_{n+1} dz_{n+1}, \quad dx = \frac{1}{u} dz_0 + \omega_{[1]} dz_1 - \frac{\delta}{u} dz_{n+1},$$

direct comparison tells us

$$x_{z_0} = \frac{1}{u}, \quad x_{z_1} = \omega_{[1]}, \quad x_{n+1} = -\frac{\delta}{u}.$$

And $\forall \omega_{[i]} = x_{z_i}, i = 2, \ldots, n$ where $x_{z_i} = \frac{\partial x}{\partial z_i}$, we have introduced the additional variables $z_2, \ldots, z_n$. which transforms the mCHH$(2+1)$ into

$$\left(\frac{x_{z_i z_0 z_0}}{x_{z_0}} + \frac{x_{z_{i+1}}}{x_{z_0}} \right)_{z_0} = \left(\frac{x_{z_0}^2}{2} \right)_{z_i}, \quad i = 1, \ldots, n.$$

which we can present in the form of n copies of a **modified CBS (mCBS)** equation

$$m_{z_0} = \frac{x_{z_0}^2}{2}, \quad m_{z_i} = \frac{x_{z_i z_0 z_0}}{x_{z_0}} + \frac{x_{z_{i+1}}}{x_{z_0}}.$$

Inherited transformation

There exists a Miura transform between the CBS and mCBS equations

$$M = \frac{x_{z_0} - m}{4}$$

From here, we can relate the dependent and independent variables as

$$\frac{1}{u} = \left(\frac{1}{P}\right)_x + \frac{1}{P},$$
$$\frac{\delta}{u} = \left(\frac{\Delta}{P}\right)_x + \frac{\Delta}{P},$$
$$\omega_{[i]} = \frac{1}{2}((\Omega_{[i]})_x + \Omega_{[i]}).$$

Or the inverse, mCHH$(2+1)$ to CHH$(2+1)$

$$\Delta = (v_{[n]})_x - v_{[n]},$$
$$P\Omega_{[i+1]} = 2(v_{[i]} - (v_{[i]})_x)$$

Conclusions

- We provided a reciprocal transformation from the CHH$(2+1)$ hierarchy containing n fields $\Omega_{[i]}$ and U fields and three independent variables (X, Y, T) to n copies of the CBS (containing one field M and $n+1$ independent variables $z_0, \ldots, z_n$), each of which is written in three different varibles z_0, z_i, z_{i+1}
- We provided a reciprocal transformation from the mCHH$(2+1)$ hierarchy with n fields $\omega_{[i]}$ and n fields $v_{[i]}$, a u field and three independent variables (x, y, t) to n copies of the mCBS equation (containing one field m and $n+1$ independent variables $z_0, \ldots, z_n$), each of which is written in three different varibles z_0, z_i, z_{i+1}.
- We have introduced the Miura transform between the CBS and mCBS equations.
- We have obtained a Miura-reciprocal transform between CHH$(2+1)$ and mCHH$(2+1)$. In this way, we have related all fields present in CHH$(2+1)$ to all fields in mCHH$(2+1)$.
- We have shown how to obtain a Lax Pair for CHH$(2+1)$ and mCHH$(2+1)$ with the aid of the inverse reciprocal transform, relying on CBS and mCBS Lax pairs.

Publications on Reciprocal Transformations

- Miura reciprocal Transformations for hierarchies in 2+1 dimensions,
  P.G. Estévez, C. Sardón,
  J. Nonl. Math. Phys. **20**, 552–564 (2013).
- Miura reciprocal transformations for two integrable hierarchies in 1+1 dimensions,
  P.G. Estevez, C. Sardón,
  Proceedings GADEIS, Protaras, Cyprus (2012),
  arXiv:1301.3636

Future endeavours

1. It would be interesting to find Lie systems with VG of Hamiltonian vector fields w.r.t. *almost or twisted Poisson structures*.
2. The generalization of the theory of Lie systems to Lie algebroids. The latter have shown their importance in Geometric Mechanics and Control theory.
3. The theory of PDEs in relation to the coalgebra method is still an open question.
4. We have plans of continuing a similar line of research applied to *k-symplectic or Kähler geometries*.
5. An interesting plan is to find reasons for the differences between the Lie symmetries of the Lax pair and the nonlinear equation itself. Expanding our investigacion to *contact symmetries* could also help in completing our problem.
6. For reciprocal transformations, the could use the trial of composition of transformations of different nature, which can lead to more unexpected but desirable results, as in our example exposed.